Mathematical Problem-Solving

Foundation Block in Mathematics, Quest University Canada
April 2014

Prerequisite:	None
Class Hours:	9:00AM to 12:00PM, in Room #315
Tutor:	Richard Hoshino, Academic Building, Office #425 (office) 604-898-8078, (cell) 604-848-5503, richard.hoshino@questu.ca
Textbook:	*The Math Olympian*, available at the Quest bookstore.
Peer Tutor:	Bailey Holderman, bailey.holderman@questu.ca

This course is about the heart of mathematics, a collection of beautiful problems connected together in unexpected ways. The problems are chosen from a wide spectrum, ranging from recreational puzzles and brain teasers to contest problems. Students will also read a math novel, in which the main character learns the art of problem-solving and through that process, develops insight, imagination, confidence, creativity, and critical thinking. Students will use this novel as a springboard to reflect upon their own mathematical journey and explore how problem-solving principles and techniques can be applied to address some of society's toughest challenges.

Course Evaluation

Problem Sets (6 × 7.5%)	**45%**
Syntheses (3 × 10%)	**30%**
Reflections (2 × 5%)	**10%**
Final Project	**15%**

There are four methods of assessment in this course.

- **Problem Sets** will consist of questions whose topics/ideas are uncovered during class. Each problem set will consist of one question to be completed individually, and two questions to be completed in groups of three. (The exception will be for Problem Set #6, which will involve a well-known challenge puzzle!)

- **Syntheses** will consist of problems connecting different areas of the course, allowing you to sum up (or *synthesize*) what you have learned. Think of them as take-home exams.

- **Reflections** will be your personal responses to questions inspired by the course, describing your problem-solving journey and how it connects to your Quest education.

- **Final Project** work will occur during the last week of the course. Each group will pose a Question inspired by a real-life issue, uncover the mathematics contained in this Question, submit a detailed written report, and deliver a 15-minute presentation on the last day of class.

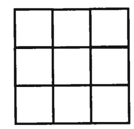
Course Description

Here's a question from a typical high school math class:

> ***Question 1***: *Mrs. Smith has received four free tickets to a Justin Bieber concert, and decides to give them away: to two of her female friends, and two of her male friends.*
> *Her female friends are Alice, Bethany, Chamique, and Diana.*
> *Her male friends are Edwin, Fernando, George, and Harry.*
> *Determine the number of different ways Mrs. Smith can give out the four tickets.*

If you did any math enrichment in high school (e.g. math contests, math clubs), you might have seen recreational puzzle questions like this:

> ***Question 2***: *Count the total number of squares and rectangles, of all sizes, that appear in a 3 by 3 unit square.*



This is not a typical math course, nor is it a block on math enrichment. In this course, you will not calculate answers to meaningless questions devoid of context or purpose; rather, you will *solve problems*, and learn how to rigorously communicate your solutions, both orally and in writing. You will discover core problem-solving principles, used by scientists in every discipline, and explore the applications of these principles to the real world, on issues that matter to you.

Through the process of doing and creating mathematics, you will learn how to think mathematically, develop your creativity and critical thinking, and enhance your skills in Rhetoric and Quantitative Reasoning. In other words, this is a mathematics course at Quest.

To give you a taste of what you'll experience, here is a problem from our course.

> *(a) Answer Question 1 above, carefully explaining each of your steps.*
>
> *(b) Answer Question 2 above, carefully explaining each of your steps.*
>
> *(c) Which question did you find harder to solve? Why?*
>
> *(d) Carefully explain why the two questions are* ***completely identical*** *(!!)*
> *(Hint: find a clever way to transform Question 1 into Question 2, and vice-versa.)*

The preceding problem illustrates a powerful problem-solving principle:

Solve a challenging problem by first converting it to an equivalent simpler problem.

During our 18 days together, you will develop the insight to recognize how a difficult problem in one context can be transformed into an equivalent yet simpler problem in another context. We will discover how Quest students applied what they learned in this course to implement a new roommate matching algorithm, as well as propose an enhanced course scheduling process. We will also explore how complex real-world issues can be transformed into simple math problems, and solved using mathematical techniques, leading to measurable and meaningful social change.

If you would like a free e-copy of this novel, please email me: richard.hoshino@gmail.com

Course Text

In March 2010, I moved to Tokyo after my wife Karen landed her dream job at a highly-regarded Japanese university. As an unemployed house-husband starting a new life in a new country, I asked the following Question: "In what ways can a student experience mathematics to develop the confidence, critical thinking, and communication skills so important in life?"

I answered my Question by writing a novel inspired by my life experiences, a 425-page Keystone Project entitled *The Math Olympian*, the story of an insecure teenager who commits herself to the crazy and unrealistic dream of representing her country at the International Mathematical Olympiad, and thanks to the support of innovative mentors, combined with her own relentless perseverance, discovers meaning, purpose, and joy.

You will read *The Math Olympian*, which describes the mathematical journey of a (fictional) teenager, as a springboard to reflect upon *your* mathematical journey. You will explore how your pre-Quest education has shaped your perception of mathematics and how it has affected your confidence in doing math. As we progress through the block, you will learn the role of mathematics in a Liberal Arts education, both in your current studies and in your concentration years where you will pursue your Question. Near the end of the course, you will imagine your future, where you will apply what you learned in this course to tackle the issues that inspire your passion so that you can be the change you wish to see in the world. You will write two Reflection pieces (500 to 750 words each), where you will describe this journey and explore the ways in which a course on mathematical problem-solving can be of service to you.

Course Forum

We will use the Course Home Page (CHP), available on Quest's PowerCampus Self-Service. Please check this website at least once each day. The URL is:

https://chp.questu.ca/sites/2014-SPRING/BLOCK4-MAT 2008-SEMR-02/

Our Course Home Page contains a private Course Forum. You will post your 500-750 word reflection mini-essays on the Course Forum. We will also have a class Google Doc for various administrative items, which I will also ask you to check at least once each day. The URL is:

https://docs.google.com/document/d/11DBRmMqTgKcaWQ2eIneriOjRMmYcD_kcMGoD4ahmgh8

In addition to posting your 500-750 word reflection, you will also comment on at least one other student's reflection – for example, you could thank the writer for sharing their work and explain why their reflection was meaningful to you; or perhaps, you want to tell the writer that you agree with their perspective and share a similar experience; or perhaps you wish to respectfully disagree with their main point and respond with your thoughts. The comments are meant to be open-ended; all I ask is that you put some effort into your comment(s), i.e., please refrain from trivial superficiality such as "Nice reflection" or "Good job". Also you are accountable for demonstrating respect and tact when using this Course Forum, for *what* you say is less important than *how* you say it. While you will not be formally evaluated on your comments, please note that the marks on your two reflections will be adversely affected if your comments are unhelpful and/or non-existent.

Course Schedule

In Week 1, we will discover powerful *problem-solving strategies* (e.g. searching for patterns, exploiting parity, drawing the right picture), and develop new modes of *mathematical communication* that go far deeper than mechanically manipulating algebraic symbols.

In Week 2, we will develop sophisticated techniques in *symmetry* and *probability*, and apply these ideas to address two real-life problems through the process of *mathematical modeling*.

In Week 3, we will establish beautiful results in *geometry* and *number theory*, to further develop the skill of communicating concisely and rigorously, both orally and through writing.

In Week 4, you will break into small groups and pose an original Question. You will then uncover all the mathematics within this Question using the problem-solving methods you've learned throughout the block, and deliver a 15-minute presentation on the last day of class.

Day	Date	Reading Due (by 9AM)	Work Due (by 8AM)	Topic in Class
1	Mon Mar 31	*p. 1-6, 426-428*		*Equivalence*
2	Tue Apr 1	*p. 7 to 44*	**Reflection 1**	*Handshakes*
3	Wed Apr 2	*p. 45 to 68*	**Problem Set 1**	*Pigeonhole Principle*
4	Thu Apr 3	*p. 69 to 91*		*Graph Colouring*
5	Fri Apr 4	*p. 92 to 115*	**Problem Set 2**	*Parity (Part 1)*
6	Mon Apr 7	*p. 116 to 133*	**Synthesis 1**	*Parity (Part 2)*
7	Tue Apr 8	*p. 134 to 173*		*Symmetry*
8	Wed Apr 9	*p. 174 to 205*	**Problem Set 3**	*Course Scheduling*
9	Thu Apr 10	*p. 206 to 239*		*Roommate Matching*
10	Fri Apr 11	*p. 240 to 257*	**Problem Set 4**	*Probability*
11	Mon Apr 14	*p. 258 to 288*	**Synthesis 2**	*Triangle Geometry*
12	Tue Apr 15	*p. 289 to 320*		*Circle Geometry*
13	Wed Apr 16	*p. 321 to 344*	**Problem Set 5**	*Number Theory*
14	Thu Apr 17	*p. 381 to 402*	**Reflection 2**	*Number Theory*
15	Mon Apr 21	*p. 403 to 425*	**Synthesis 3**	*Final Project Work*
16	Tue Apr 22		**Problem Set 6**	*Final Project Work*
17	Wed Apr 23			*Final Project Work*
18	Thu Apr 24		**Final Project Report**	*Final Presentations*

I have designed the course so that you will be doing approximately five hours of out-of-class work every day from Monday to Friday, as well as five hours of homework during the weekend. This homework includes the time you'll require to read the course novel, *The Math Olympian*. How you structure your weekend study time (e.g. 3 hours on Saturday, 2 on Sunday) is up to you.

By Day 17, you will know your marks on all six Problem Sets (45%), all three Syntheses (30%), and both Reflections (10%). Thus, 85% of your final mark will be known to you prior to the Final Project report and presentation on the final day. By the afternoon of Day 18, I will have evaluated your Final Project (15%), and will post your final grade on the Course Home Page.

Mathematical Problem-Solving

REFLECTION #1

Due at 8AM on Tuesday April 1st

As explained in class, you will write two reflections in this course, and post each reflection on our private Course Forum.

In addition to posting your reflection, you must also <u>comment</u> on at least one other student's reflection - for example, you could thank the writer for sharing their work and explain why their reflection was meaningful to you; or perhaps, you want to tell the writer that you agree with their perspective and share a similar experience; or perhaps you wish to respectfully disagree with their main point and respond with your thoughts.

The comments are meant to be open-ended; all I ask is that you put some effort into your comment(s), i.e., please refrain from trivial superficiality such as "Nice reflection" or "Good job". Also you are accountable for demonstrating respect and tact when using the Course Forum, for *what* you say is less important than *how* you say it.

Your reflection must be posted on the Course Forum by 8AM on Tuesday April 1st. You must comment on at least one post by 8AM on Thursday April 3rd.

Each reflection is worth 5% of your grade. Your post must be between 500 and 750 words. In evaluating your reflection, I will follow a rubric used by Quest's former Rhetoric Coordinator.

Reflection Question #1

Describe the highlights and lowlights of your mathematics education prior to your arrival at Quest, and explain how these experiences have shaped your confidence in doing mathematics.

Here are some questions that may be helpful to you as you write your reflection:

(a) What aspects about math do you like/dislike?

(b) Which of your math experiences have been especially easy/difficult?

(c) Were you inspired and/or empowered thanks to an excellent math teacher?

(d) Were you disempowered and/or traumatized because of a terrible math teacher?

Assigned Groups for the Problem Sets

Here are your assigned groups for the five Problem Sets. Over the course of the block, you will work with each of your classmates exactly once. This mathematically-optimal arrangement, known as a Steiner Triple System, is yet another application of mathematical problem-solving!

Problem Set 1 Question #1 (due Apr 2nd)	Zach RW Zach K Billy	Net Ferin Peter	Mikaila James Xaviera	Lucy Ian Frenchie	Grace Noah Andrew	Katelyn Dominique Rryla	Becca Byron Kyle
Problem Set 1 Question #2 (due Apr 2nd)	Mikaila Net Noah	Katelyn Lucy James	Zach K Ian Dominique	Zach RW Byron Xaviera	Grace Becca Rryla	Billy Peter Kyle	Ferin Frenchie Andrew
Problem Set 2 Question #1 (due Apr 4th)	Zach RW Katelyn Becca	Noah James Dominique	Net Lucy Byron	Grace Ian Xaviera	Zach K Ferin Kyle	Mikaila Peter Andrew	Billy Frenchie Rryla
Problem Set 2 Question #2 (due Apr 4th)	Zach K Grace Mikaila	Billy Katelyn Ian	Net Becca Frenchie	Ferin Dominique Byron	Zach RW James Andrew	Lucy Peter Rryla	Noah Xaviera Kyle
Problem Set 3 Question #1 (due Apr 9th)	Net Billy James	Lucy Ferin Becca	Grace Dominique Peter	Zach K Xaviera Frenchie	Zach RW Noah Rryla	Mikaila Ian Kyle	Katelyn Byron Andrew
Problem Set 3 Question #2 (due Apr 9th)	Zach K Lucy Noah	James Becca Peter	Billy Ferin Xaviera	Zach RW Dominique Frenchie	Grace Katelyn Kyle	Net Ian Andrew	Mikaila Byron Rryla
Problem Set 4 Question #1 (due Apr 11th)	Grace Zach RW Net	Mikaila Lucy Dominique	Billy Noah Byron	Katelyn Peter Xaviera	Zach K Becca Andrew	Ferin Ian Rryla	James Frenchie Kyle
Problem Set 4 Question #2 (due Apr 11th)	Billy Mikaila Becca	Ferin Katelyn Noah	Zach RW Peter Ian	Grace Frenchie Byron	Zach K James Rryla	Net Dominique Kyle	Lucy Xaviera Andrew
Problem Set 5 Question #1 (due Apr 16th)	Grace Ferin James	Becca Noah Ian	Mikaila Katelyn Frenchie	Zach K Peter Byron	Zach RW Lucy Kyle	Billy Dominique Andrew	Net Xaviera Rryla
Problem Set 5 Question #2 (due Apr 16th)	Kyle Andrew Rryla	Zach RW Mikaila Ferin	Zach K Net Katelyn	Grace Billy Lucy	Noah Peter Frenchie	James Ian Byron	Becca Dominique Xaviera

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PROBLEM SET #1

Due at 8AM on Wednesday April 2^{nd}

Each Problem Set will be marked out of 15, and is worth 7.5% of the final course grade. There are three problems, each worth five marks.

Problems #1 and #2 must be completed in groups of three; you are to submit just one solution per group. You are strongly encouraged, though not required, to have a different "scribe" for each of parts (a), (b), and (c).

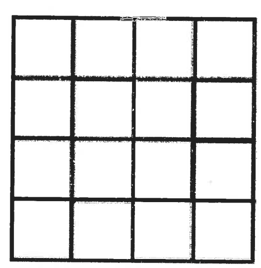
Hand in the solutions to each of the three problems on *separate* sheets of paper, with your name(s) at the top of your solution. For each problem, if your solution requires more than one page, staple those pages together. Please type or neatly handwrite your solutions on standard 8.5 × 11 paper.

While a solution must be absolutely perfect to receive full marks, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress.

If you are stuck on the problems, I encourage you to contact Haley, the subject-specialist peer tutor for this course: either on your own or in a small group.

Question #1 - Group Problem

(a) Determine the number of squares and rectangles, of all sizes, that appear in a 4 by 4 unit square.



(b) Solve the above problem by converting it to the Ticket Problem, which is both equivalent and simpler. (Make sure you clearly justify why the two problems are identical.)

(c) Describe a real-life question (or societal issue) that you think might be reducible to a simpler equivalent math problem. Explain.

Question #2 - Group Problem

Mr. and Mrs. Collins attended a charity fundraiser and sat at a table with three other married couples. Various handshakes took place among the eight guests.

No one shook hands with his/her partner (and of course, no one shook their own hand!)

After all the introductions had been made, Mr. Collins asked each of the other seven people how many hands they shook. Surprisingly, they all gave a different answer.

(a) Explain why someone at the party must have shaken 6 hands, and why someone at the party must have shaken 0 hands. Explain why these two people must be married.

(b) Explain why the person who shook five hands must be married to the person who shook one hand.

(c) How many hands did Mr. Collins shake? Justify your answer.

Question #3 - Individual Problem

Solve the following "cross-number" puzzle by putting the proper digit into each box. All answers to the clues are three-digit numbers (no answer begins with zero).

1	2	3
4		
5		

ACROSS

1. A prime number
4. Not a prime number
5. A perfect square

DOWN

1. A power of 5 (i.e., $5^1, 5^2, 5^3, 5^4, \ldots$)
2. A power of 2
3. A power of 3

In your solution, explain the process by which you solved this cross-number puzzle. (Make sure you clearly justify why your completed grid is the only possible solution.)

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PROBLEM SET #2

Due at 8AM on Friday April 4^{th}

Each Problem Set will be marked out of 15, and is worth 7.5% of the final course grade.

Hand in the solutions to each of the three problems on *separate* sheets of paper, with your name(s) at the top of your solution. For each problem, if your solution requires more than one page, staple those pages together. Please type or neatly handwrite your solutions on standard 8.5×11 paper.

While a solution must be absolutely perfect to receive full marks, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress.

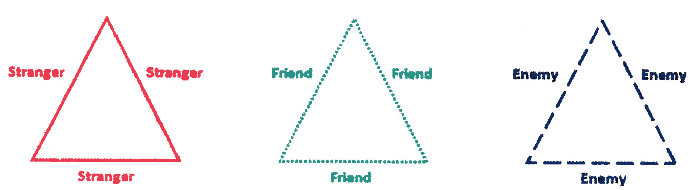
If you are stuck on the problems, I encourage you to contact Haley, the subject-specialist peer tutor for this course: either on your own or in a small group.

Question #1 - Group Problem

Some Quest students have Facebook accounts. For each pair of students, exactly one of the following statements is true: "they haven't met", "they like each other", or "they hate each other".

(a) Suppose there are 17 Questers on Facebook, and Rryla is one of them. Explain why at least one of these three statements must be true in the 17-vertex Facebook graph: (i) Six people are friends with Rryla; (ii) six people are strangers with Rryla; (iii) six people are enemies with Rryla.

(b) If there are 17 Questers on Facebook, prove that there must exist three people from this group, all of whom are either mutual strangers, mutual friends, or mutual enemies.

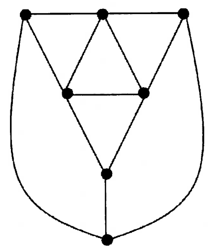


(c) If there are only 10 Questers on Facebook, show that the above conclusion does <u>not</u> necessarily hold. To do this, construct a graph on 10 vertices, and colour each edge with one of three colours so that there is no monochromatic triangle, i.e., a triangle with all three edges the same colour. Make sure you clearly explain the process by which you obtained your edge colouring.

(d) (**BONUS**) If there are exactly 16 Questers on Facebook, must there be three people from this group, all of whom are either mutual strangers, mutual friends, or mutual enemies?

Question #2 - Group Problem

(a) We say that a graph is *3-colourable* if each of its vertices can be coloured with one of three colours (red, blue, green) so that there is no edge connecting two vertices of the same colour. Carefully explain why the graph below, with 7 vertices and 12 edges, is not 3-colourable.



(b) Alexis, Becky, Courtney, Daniella, Emily, Fiona, Gabby are seven first-year students who all wish to take a summer block in May. (There are exactly three course options for the May block.) Below is the list of courses these students have taken, or will take, from January to April.

Team	January	February	March	April
Alexis	Reason/Freedom	Problem-Solving	Spanish	Neurobiology
Becky	Reason/Freedom	Chinese	World Religions	Global Perspectives
Courtney	Identity/Perspective	Chinese	Genetics in Society	Statistics
Daniella	Astrophysics	Democracy/Justice	Spanish	Global Perspectives
Emily	Identity/Perspective	Democracy/Justice	World Religions	Political Economy
Fiona	Astrophysics	Fate/Virtue	Chemistry	Political Economy
Gabby	French	Fate/Virtue	Genetics in Society	Neurobiology

The seven females are best of friends; however, they want to maximize their Quest Experience by having classes with as many different people as possible. Specifically, if two of them share a class anytime between January and April, they refuse to take the same summer block in May. (For example, Alexis and Becky won't sign up for the same class in May, as they've already taken a class together in January.) Is it possible for the seven females to select their summer course this way, or *must* some pair of students be forced to take another class together in May?

(c) Suppose there are four course options in May. Then can the seven females achieve their desired objective? If so, explain how.

Question #3 - Individual Problem

In class, we played the SIM Game. Recall the rules of the game: there are six dots ("vertices") on the board, forming a hexagon. Two players take turns joining any pair of vertices, colouring that line using a red marker (first player) or a blue marker (second player). The first player who creates a monochromatic triangle, i.e., a triangle with three edges of the same colour, loses the game.

Carefully explain why this game cannot end in a tie.

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SYNTHESIS #1

Due at 10AM on Monday April 7^{th}

This synthesis will be marked out of 20, and is worth 10% of the final course grade.

Answer four of the following eight problems. Hand in your four solutions on separate sheets of paper, with your name at the top of each solution. For each problem, if your solution requires more than one page, staple those pages together. Type or neatly handwrite your solutions on 8.5×11 paper.

Do not be stressed if you cannot complete four problems; solving the majority of them is perfectly satisfactory, and I will be generous in awarding partial marks for incomplete solutions that demonstrate progress. However, you must write a complete solution, with no gaps or holes, in order to receive full marks.

Note that your synthesis must be done *strictly individually.* No consultation with fellow students is permitted, and you may not use the internet. Instead, I strongly encourage you to refer to your class notes, as well as *The Math Olympian*, as you work on these problems.

Question #1 (5 marks)

On the first day of Mathematical Problem-Solving, each of the 12 female students in the class shook hands with every other female in the class (but they ignore the dudes, since they're all #uncool.)

Give three different proofs justifying why there must have been exactly 66 total handshakes.

Question #2 (5 marks)

Suppose there are 200 first-year students at Quest, and all have been given their own individual locker. The students notice that the lockers are all initially closed, and conspire to play a game.

The first student, the one with Locker 1, goes down the hallway and opens each of the lockers. After she's done, the second student, the one with Locker 2, goes down the hallway and closes all the lockers that are multiples of 2. After he's done, the third student, the one with Locker 3, changes the state of all lockers that are multiples of 3, closing the open lockers and opening the closed lockers. And this continues, with each student going down the hallway one after the other, altering the lockers whose numbers are multiples of their locker number.

At the end, how many lockers are open, and which ones are they? Clearly justify your answer.

Question #3 (5 marks)

Each of the integers 346 and 662 have digits whose product is 72. How many three-digit positive integers have digits whose product is 72? Clearly justify your answer.

Question #4 (5 marks)

Lucy writes down the eleven numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ on the board. She plays the following game. She randomly selects two numbers, x and y (with $x < y$), erases them from the board, and writes down their difference $y - x$.

She keeps repeating this until only one number remains on the board. Clearly explain why this final number *must* be odd.

p.s. this problem is *identical* to a problem we solved in class today. Why?

Question #5 (5 marks)

Let n be a positive integer. For all positive divisors d of n^2, write down the fraction $\frac{n}{n+d}$, and add up all these fractions to create a "magic sum".

For example, if $n = 3$, then the divisors of $n^2 = 9$ are $\{1, 3, 9\}$, and the magic sum is

$$\frac{3}{3+1} + \frac{3}{3+3} + \frac{3}{3+9} = \frac{3}{2}.$$

For example, if $n = 4$, then the divisors of $n^2 = 16$ are $\{1, 2, 4, 8, 16\}$, and the magic sum is

$$\frac{4}{4+1} + \frac{4}{4+2} + \frac{4}{4+4} + \frac{4}{4+8} + \frac{4}{4+16} = \frac{5}{2}.$$

Determine the magic sum for $n = 5$, $n = 6$, $n = 8$, and $n = 10$. Can you find a pattern that enables you to quickly calculate the magic sum for these values of n?

For a bonus mark, determine the magic sum for $n = 100$.

Question #6 (5 marks)

Five tutors at Quest (Andre, Bob, Court, Darcy, Eric) have five different offices (401, 402, 403, 404, 405). Each tutor has a different way of coming to school (bicycle, walk, car, bus, taxi), arriving on campus at five different times (7:00, 7:15, 7:30, 7:45, 8:00).

You are given the following clues:

(a) Bob is not in 401.
(b) The tutor in 404 does not own a bicycle.
(c) The tutor who arrives at 8:00 is the one who rides his bicycle.
(d) The tutor in 402 arrives later than Eric but before the tutor who walks to work.
(e) Darcy, the tutor in 403, arrives at 7:45 each morning.
(f) Neither Court nor the tutor in 404 comes in at 7:00, and neither tutor owns a car.
(g) The tutor in 401, who is not Eric, takes the bus. He always arrives fifteen minutes before the tutor who arrives via taxi.

From the above clues, determine each tutor's office number, mode of transportation and the time they arrive at Quest. Clearly describe the process by which you arrived at your answer.

NOTE: the remaining questions (#7, 8) were a 4-star sudoku and a 6-star sudoku. (thus, they weren't included in this package).

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PROBLEM SET #3

Due at 8AM on Wednesday April 9th

Each Problem Set will be marked out of 15, and is worth 7.5% of the final course grade.

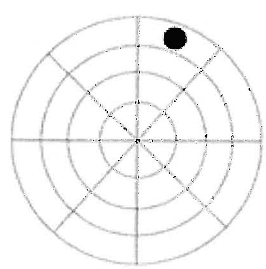
Hand in the solutions to each of the three problems on *separate* sheets of paper, with your name(s) at the top of your solution. For each problem, if your solution requires more than one page, staple those pages together. Please type or neatly handwrite your solutions on standard 8.5×11 paper.

While a solution must be absolutely perfect to receive full marks, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress.

If you are stuck on the problems, I encourage you to contact Bailey, the subject-specialist peer tutor for this course: either on your own or in a small group.

Question #1 - Group Problem

Ferin and Ian play a game by alternately moving a hockey puck on a board with n concentric circles divided into r regions. For example, in the diagram below, we have $n = 4$ and $r = 8$.



The game starts with the puck already on the board, as shown. A player may move either clockwise one position or one position towards the centre, but cannot move to a position that has been previously occupied. The last person who is able to move wins the game. Ferin moves first.

(a) If $n = 4$ and $r = 8$, show that Ferin has a winning strategy (i.e., explain how Ferin can always win the game no matter how well Ian plays).

(b) If $n = 3$ and $r = 9$, show that Ian has a winning strategy (i.e., explain how Ian can always win the game no matter how well Ferin plays).

(c) If $n = 4$ and $r = 9$, which player has a winning strategy? Explain.

Question #2 - Group Problem

Grace and Noah play the following two-player game: at the beginning, there are a bunch of coins on a table, and on each turn, a player can either remove *one* coin or remove *two* coins.

The goal of the game is to be the person who removes the last coin, and that person is declared the winner. Suppose Grace always moves first.

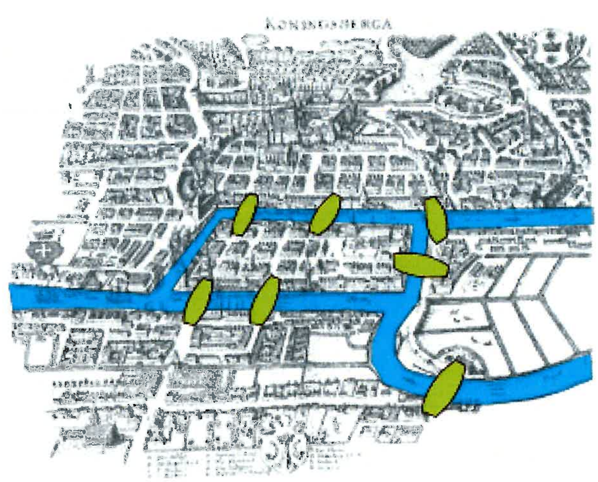
(a) If there are 10 coins at the start, which player has a winning strategy? Describe that strategy.

(b) If there are 15 coins at the start, which player has a winning strategy? Describe that strategy.

(c) If there are 20 coins at the start, which player has a winning strategy? Describe that strategy.

Question #3 - Individual Problem

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of a river, and included two large islands which were connected to each other and the mainland by seven bridges.



Is it possible to walk through the city and cross each bridge exactly once?

NOTE: The islands cannot be reached by any route other than the bridges, and every bridge must be crossed completely every time; one cannot walk halfway onto the bridge and then turn around and later cross the other half from the other side.

Mathematical Problem-Solving

PROBLEM SET #4

Due at 8AM on Friday April 11th

Each Problem Set will be marked out of 15, and is worth 7.5% of the final course grade.

Hand in the solutions to each of the three problems on *separate* sheets of paper, with your name(s) at the top of your solution. For each problem, if your solution requires more than one page, staple those pages together. Please type or neatly handwrite your solutions on standard 8.5 × 11 paper.

While a solution must be absolutely perfect to receive full marks, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress.

If you are stuck on the problems, I encourage you to contact Bailey, the subject-specialist peer tutor for this course: either on your own or in a small group.

Question #1 - Group Problem

You have been hired by Minute Maid to re-design the shape of their mini juice boxes.

Currently, each juice box has width 2cm, length 4cm, and height 8cm. The company president has stipulated that the juice box must have length 4cm and volume 64cm^3, but that the width and height can vary according to your recommendations.

(a) Suppose the width of the juice box is x cm. Prove that the total surface area of the juice box is $8x + \frac{128}{x} + 32$.

(b) You have decided that the best way to design the juice box is to minimize the total surface area. The company president has asked you to decide the width and height of their juice box. What are your recommendations? (Feel free to use an Excel spreadsheet to justify your answer.)

(c) Suppose Minute Maid spends one million dollars each year just to purchase the cardboard that goes into making these mini juice boxes. How much money would they save in cardboard costs by switching the shape of the juice box to your "mathematically-optimal" recommendation?

Question #2 - Group Problem

In class today, we discovered the Gale-Shapley Stable Marriage Algorithm, in which n women are matched to a unique man, thus producing n stable marriages.

Each man ranks the n women in order of preference, from most desirable to least desirable. Similarly, each woman ranks the n men in order of preference, from most desirable to least desirable.

In the first round, each woman proposes to the man she prefers the most, and then each man replies "maybe" to the woman he most prefers (among all the women from whom he has received a proposal), and rejects the rest. He is then tentatively engaged to this woman.

In each subsequent round, each unengaged woman proposes to the most-preferred man on her preference list to whom she has not yet proposed, regardless of whether he is currently engaged.

Then each man replies "maybe" to the woman he most prefers, and rejects the rest. He is able to "trade up" by jilting his current partner for a new woman if he receives a proposal from someone higher on his preference list.

The process ends when no woman is unengaged. Then each man turns to the woman with whom he is tentatively partnered, and says, "I do." Then these n couples live happily ever after.

(a) If $n = 5$, explain why the algorithm lasts at most five rounds, with at most twenty-five total proposals.

(b) Explain why the algorithm must end with every woman and every man getting married.

(c) We say that a woman and a man form a *blocking pair* if they are not married, but prefer each other over the person with whom they are married. If no blocking pairs exist in the final matching, we say that the matching is *stable*. Prove that the final output of the Gale-Shapley Algorithm must be a stable matching.

Question #3 - Individual Problem

Frenchie and James play the following game on a large *circular* table. They take turns placing a quarter anywhere on the table, with the rule that this quarter cannot touch any other quarter already on the table.

Eventually one person will not be able to place a quarter anywhere on the table, in which case he will lose.

Suppose that Frenchie moves first, and that there are infinitely many quarters.

Does either player have a winning strategy? If so, explain who can always win, and how.

Mathematical Problem-Solving

SYNTHESIS #2

Due at 8AM on Monday April 14th

This synthesis will be marked out of 20, and is worth 10% of the final course grade.

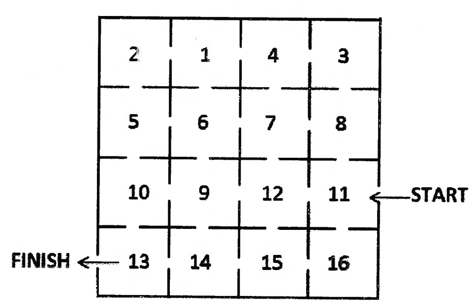
Answer four of the following nine problems. Hand in your four solutions on separate sheets of paper, with your name at the top of each solution. For each problem, if your solution requires more than one page, staple those pages together. Type or neatly handwrite your solutions on 8.5×11 paper.

Do not be stressed if you cannot complete four problems; solving the majority of them is perfectly satisfactory, and I will be generous in awarding partial marks for incomplete solutions that demonstrate progress. However, you must write a complete solution, with no gaps or holes, in order to receive full marks.

Note that your synthesis must be done *strictly individually.* No consultation with fellow students is permitted, and you may not use the internet. Instead, I strongly encourage you to refer to your class notes, as well as *The Math Olympian*, as you work on these problems.

Question #1 (5 marks)

Each of 16 rooms has a "point value", as indicated in the diagram below. Your team's task is to collect as many points as possible. Enter at the start, exit at the finish, and pass through each room no more than once. What is the greatest number of points your team can collect?



Question #2 (5 marks)

(a) Zach looks at his watch at exactly 9:04AM. What is the measure of the angle formed by the hour hand and minute hand?

(b) Zach's watch is programmed so that a beep goes off every time there is a 180° angle formed by the hour hand and minute hand. In one 24-hour day, how many times does Zach's watch beep?

Question #3 (5 marks)

Determine the minimum perimeter of a triangle with one vertex at $(7, 1)$, one vertex on the x-axis, and one vertex on the line $y = x$.

(Hint: reflect the point $(7, 1)$ about the x-axis to create a new point, and also reflect $(7, 1)$ about the line $y = x$ to create another new point. How might this information be of help?)

Question #4 (5 marks)

Suppose four women (A, B, C, D) propose to four men (E, F, G, H).

(a) Arrange the preference lists for the four women and the four men so that the Gale-Shapley Algorithm produces a stable matching in which every woman is matched to her first choice and every man is matched to his last choice. Clearly detail the steps of the Gale-Shapley algorithm (i.e., the order in which the proposals take place.)

(b) Show that for *any* arrangement of the preference lists, at most one woman can be matched to the person appearing last on her preference list, and use this result to justify that there is no arrangement of the preference lists for which more than 13 proposals take place. Is it possible to arrange the preference lists so that exactly 13 proposals occur? If so, describe how, clearly detailing the steps of the Gale-Shapley algorithm (i.e., the order in which the proposals take place.)

Question #5 (5 marks)

I throw a sequence of three darts at a dartboard, aiming for the centre.

Assume each one of my throws is equally skillful, i.e., I don't have a tendency to get better (or worse) with each throw.

(a) Suppose my second throw is better than my first. What is the probability that my third throw will be better than my second?

(b) Is this just the Car-Goat-Goat problem in disguise? If so, carefully explain why these two problems are identical. If not, explain how they are different.

NOTE: the remaining questions (#6, 7, 8, 9) were well-known puzzles: a 5-star Sudoku, a 6-star Sudoku, a 6×6 Kenken, and a 8×8 Kenken.

Mathematical Problem-Solving

PROBLEM SET #5

Due at 8AM on Wednesday April 16th

Each Problem Set will be marked out of 15, and is worth 7.5% of the final course grade.

Hand in the solutions to each of the three problems on *separate* sheets of paper, with your name(s) at the top of your solution. For each problem, if your solution requires more than one page, staple those pages together. Please type or neatly handwrite your solutions on standard 8.5 × 11 paper.

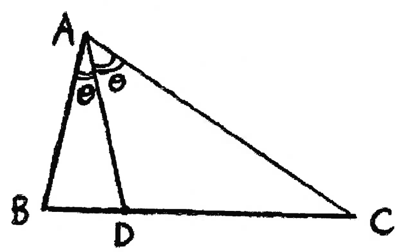
While a solution must be absolutely perfect to receive full marks, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress.

If you are stuck on the problems, I encourage you to contact Bailey, the subject-specialist peer tutor for this course: either on your own or in a small group.

Question #1 - Group Problem

Take any triangle ABC, and let AD be the *internal angle bisector* of $\angle A$.

Develop <u>three</u> different proofs to the Internal Angle Bisector theorem: $\dfrac{AB}{AC} = \dfrac{BD}{DC}$.



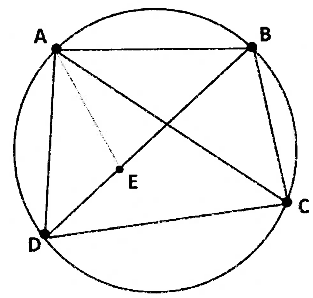
(a) Construct point P on AB so that DP is perpendicular to AB. Construct point Q on AC so that DQ is perpendicular to AC. Calculate the ratio of the area of $\triangle ABD$ to the area of $\triangle ACD$. By calculating this ratio in two different ways, conclude that $\frac{AB}{AC} = \frac{BD}{DC}$.

(b) Extend line AD to the point E so that $AC = CE$. By finding two pairs of similar triangles, and using the fact that $CE = AC$, conclude that $\frac{AB}{AC} = \frac{BD}{DC}$.

(c) Extend line BA to the point F so that $AF = AC$. By showing that AD is parallel to FC, and using the fact that $AF = AC$, conclude that $\frac{AB}{AC} = \frac{BD}{DC}$.

Question #2 - Group Problem

Let $ABCD$ be a cyclic quadrilateral. Construct point E on BD such that $\angle AED = \angle ABC$.



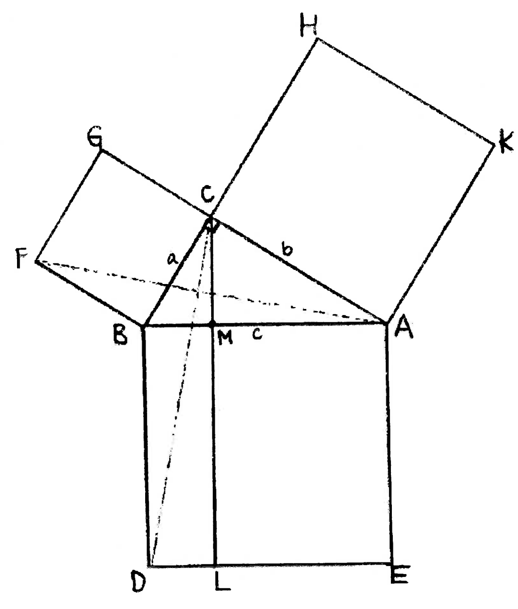
(a) Carefully explain why $\triangle AED$ is similar to $\triangle ABC$ and why $\triangle AEB$ is similar to $\triangle ADC$.

(b) Use the above two results to prove Ptolemy's Theorem: $AD \times BC + AB \times CD = AC \times BD$.

(c) Apply Ptolemy's Theorem to prove the Pythagorean Theorem.

Question #3 - Individual Problem

Prove the Pythagorean Theorem: $a^2 + b^2 = c^2$.



Mathematical Problem-Solving

REFLECTION #2

Due at 8AM on Thursday April 17th

As explained in class, you will write two reflections in this course, and post each reflection on our private Course Forum.

In addition to posting your reflection, you must also <u>comment</u> on at least one other student's reflection - for example, you could thank the writer for sharing their work and explain why their reflection was meaningful to you; or perhaps, you want to tell the writer that you agree with their perspective and share a similar experience; or perhaps you wish to respectfully disagree with their main point and respond with your thoughts.

Your reflection must be posted on the Course Forum by 8AM on Thursday April 17th. You must comment on at least one post by 8AM on Saturday April 19th.

Each reflection is worth 5% of your grade. Your post must be between 500 and 750 words. In evaluating your reflection, I will follow a rubric used by Quest's former Rhetoric Coordinator.

Reflection Question #2

Describe an issue that lights a fire in your heart, and explain how mathematics can play a role in addressing some aspect of this issue.

NOTE: the "issue" that you pick may be one at Quest, or in your community, or in your country, or in our world - the choice is up to you!

Here are some questions that may be helpful to you as you write your reflection:

(a) What societal issues and/or problems stir up deep emotions inside you? Why?

(b) If you have an idea for your Question, might it have a mathematical component?

(c) Think about Bethany's experiences in *The Math Olympian*. In what ways has she learned how mathematics can be applied to address real-world issues?

(d) Think about the mathematics you have done so far in this course. How have the past three weeks opened up your mind (and heart) to the beauty and power of mathematics?

(e) Think about the problem-solving strategies you have developed so far in this course. Are any of these strategies applicable or relevant to your life, on the issues that matter to you?

Problem Set #6

To be demonstrated at 11AM, on Tuesday April 22nd

As mentioned on Day 2 of the class, the sixth and final Problem Set of the class will have a single question: Solve the Rubik's Cube. At 11AM on Day 16 (April 22nd), I will jumble your cube.

You will have one hour to complete the challenge. This Problem Set will be marked out of 15, and be worth 7.5% of your final mark. Here is my Rubric for the Rubik's Cube.

Step 1 – Solve the White Cross
(of course, you can start with any colour face you want)

4 out of 15 if you complete this step

Step 2 – Solve the White Corners to complete the First Layer

8 out of 15 if you complete this step

Step 3 – Solve the Middle Layer

10 out of 15 if you complete this step

Step 4 – Yellow Cross on the Top Layer

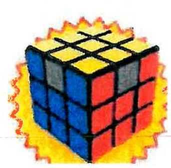
12 out of 15 if you complete this step

Step 5 – All Yellow on the Top

13 out of 15 if you complete this step

Step 6 – Yellow Corners Positioned Correctly

14 out of 15 if you complete this step



Step 7 – Yellow Edges Positioned Correctly (SOLVED!)

15 out of 15 if you complete this step and solve the cube!

Mathematical Problem-Solving

SYNTHESIS #3

Due at 8AM on Monday April 21st

This synthesis will be marked out of 20, and is worth 10% of the final course grade.

Answer two of the following six problems (10 marks). In addition, you will submit a Final Project Proposal (10 marks) as a group.

Hand in your solutions on separate sheets of paper, with your name at the top of each solution. For each problem, if your solution requires more than one page, staple those pages together. Type or neatly handwrite your solutions on 8.5×11 paper.

As always, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress. However, you must write a complete solution, with no gaps or holes, in order to receive full marks.

Excluding the Final Project Proposal, your synthesis must be done *strictly individually.* No consultation with fellow students is permitted, and you may not use the internet.

Final Project Proposal (10 marks)

In your Final Project Group, prepare a written proposal (between 800 and 1200 words), which must contain each of the following five items:

(a) A clearly defined Question that you wish to answer.

(b) A context that will help the reader understand your topic, and why this issue is important to some community that you are part of (e.g. Quest, Squamish, British Columbia, Canada, North America, or the World).

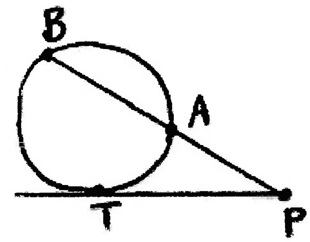
(c) An explanation of why this topic is personally relevant to each of you: include one paragraph for each person in your group.

(d) A clearly defined scope for your project: given the short time frame of this final project, what do you intend to examine, and what do you intend *not* to examine?

(e) A description of what your team plans to do to complete your work on Monday, Tuesday, and Wednesday next week.

Question #1 (5 marks)

In the diagram below, PT is a tangent to the circle, and PAB is a secant (i.e., the points P, A, and B lie on a straight line.)



(a) Prove the Tangent-Chord Theorem (that $\angle ATP = \angle ABT$) and apply it to show that $\triangle APT$ is similar to $\triangle TPB$.

(b) Prove the Tangent-Secant Theorem: $PT^2 = PA \times PB$.

(c) Apply the Tangent-Secant Theorem to prove the Pythagorean Theorem.
Hint: draw the right diagram so that $PT^2 = PA \times PB$ implies $b^2 = (c - a) \times (c + a)$.

Question #2 (6 marks)

A team of six men play the following game. They enter a room, one at a time, in any order they wish, until all six people have gone in once. In the room, there is a table with a row of six identical boxes labelled 1, 2, 3, 4, 5, 6. Inside each box is the name of one person, randomly chosen so that each person's name appears in exactly one box.

Once inside the room, each person is allowed to open and look inside 3 boxes of his choice. After each man has opened his three boxes he must leave the room, making sure that the boxes are in exactly the same state as he found them, and he is not allowed to communicate with anyone else. The team shares a million dollars if every single person manages to find the box containing his name.

The six men are allowed to get together before the game, to work out a strategy. For example, if the strategy is for everyone to guess randomly, then the probability of the team winning the million dollars is $(\frac{3}{6})^6 = (\frac{1}{2})^6 = \frac{1}{64}$, i.e., less than two percent. That's not very good.

(a) Suppose the team's strategy is as follows: three men open just the odd-numbered boxes, and the other three men open just the even-numbered boxes. Determine the probability that the team wins the million dollars.

(b) Find a strategy where the team has a winning probability exceeding 38%. Carefully describe the strategy, and calculate the probability that the team wins the million dollars.

Question #3 (6 marks)

Suppose we play the Red Card Black Card Problem with $n = 15$ people.

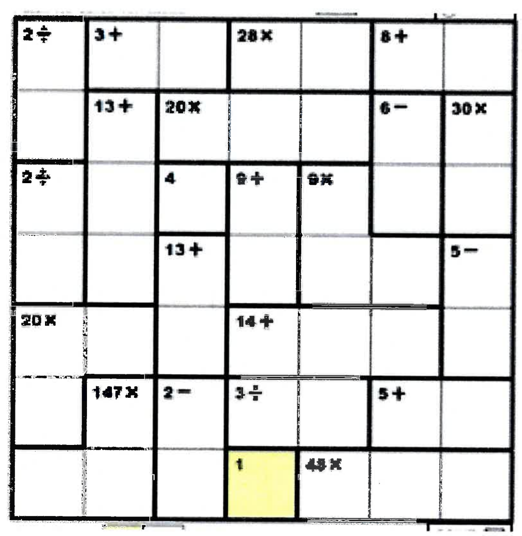
(a) Describe the optimal strategy, step by step, and apply it to the scenario where players 1, 5, 9, and 14 have red cards, while everyone else has a black card. Clearly explain what happens in this scenario. Who guesses and who passes? Does the team win?

(b) Now suppose that players 1, 2, 3, 5, 10, and 15 have red cards, while everyone else has a black card. Clearly explain what happens in this scenario. Who guesses and who passes? Does the team win?

(c) Prove that the 15-person team wins the game with probability $\frac{15}{16}$, assuming everybody is aware of the optimal strategy and follows it without error.

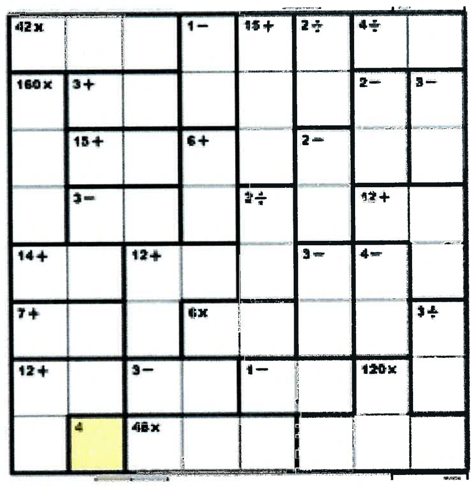
Question #4 (5 marks)

Solve this 7×7 Kenken puzzle.
NOTE: If you answer Question #4, you cannot answer Questions #5 or #6.



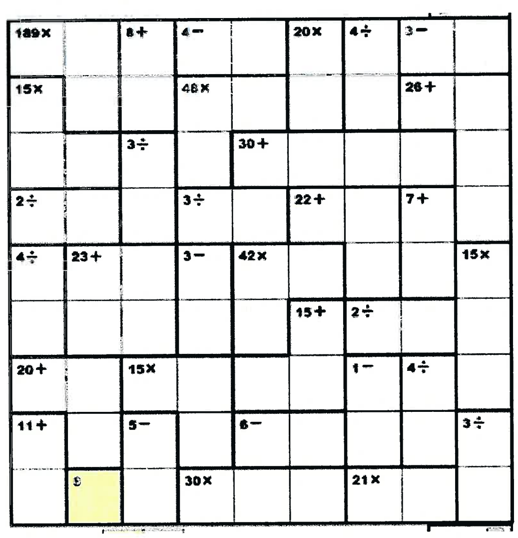
Question #5 (5.5 marks)

Solve this 8 × 8 Kenken puzzle.
NOTE: If you answer Question #5, you cannot answer Questions #4 or #6.



Question #6 (6 marks)

Solve this 9 × 9 Kenken puzzle.
NOTE: If you answer Question #6, you cannot answer Questions #4 or #5.



Mathematical Problem-Solving

Final Project

You will end this course with a final project, where you will put together the tools and techniques you have learned in order to answer a Question that is important to you.

Over the next few days, I encourage you to post potential Questions on the Google Doc. On Friday, we will select six Questions.

The 21 of you will be assigned to one Question, and each Question will form a Final Project group: either a three-person group or a four-person group.

This final assessment will consist of a project proposal (due on Monday April 21st), a final report (due on Thursday April 24th), and a presentation (on Thursday April 24th).

- Project Proposal (due April 21st), which will be part of your Weekend Synthesis.
- Final Project Report (due April 24th), worth 10% of your final course grade.
- Final Presentation (due April 24th), worth 5% of your final course grade.

So that there is no ambiguity, here is how the Final Project Report will be marked:

- **Introduction (4 marks):** a context that will help me understand your topic, with a clearly-defined Question inspired by your issue with a rationale that explains what you are doing and why. I'd also like a couple of paragraphs where you tell me why this particular issue is important to each of you, and why you are personally invested in answering this question.
- **Analysis (12 marks):** a description of the methods you used to gather your data and/or solve your problem. What did you do and why? Show clear steps throughout every step of your analysis. I recognize that each report will be different, so I will customize these 12 marks to align with the specific nature of your project.
- **Conclusion (4 marks):** based on your analysis, answer your Question. Then discuss the weaknesses and limitations of your project, and suggest avenues for future research. And finally, conclude with a paragraph describing what you learned from this project, and whether this report will be of any value to you – either for your possible Keystone, or for some other project or endeavour you wish to pursue in the future.